Consistent definitions of flux and the dual superconductivity parameters in $SU(2)$ lattice gauge theory

Richard W. Haymaker*

Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA

Takayuki Matsuki†

Tokyo Kasei University, 1-18-1 Kaga Itabashi, Tokyo 173-8602, Japan

(Dated: May 18, 2005)

Abstract

We revisit the confinement problem in maximal Abelian gauge $SU(2)$ gluodynamics as a dual superconductor through the study of the dual Abrikosov vortex. There are three effects that have not been included in previous studies. We employ a definition of flux that satisfies the exact Ward-Takahashi identity giving exact electric Maxwell equations for lattice averages. Second we modify the standard definition of magnetic current to give consistent magnetic Maxwell equations. Finally we point out that the dual Ginzburg-Landau-Higgs model is an oversimplification of the physics of the system because of the presence of significant electric currents. As a result we need a third parameter to describe the vortex in addition to the standard ones, i.e., the London penetration depth and the coherence length. Without a complete model at our disposal, we estimate the values of these three parameters for $\beta = 2.5115$ on a 32^4 lattice. As a digression, we also show that the truncation of monopoles to the percolating cluster has only a minor effect on the vortex profile.

PACS numbers: 11.15.Ha, 12.38.Aw, 12.38.Gc

*Electronic address: `haymaker@rouge.phys.lsu.edu`

†Electronic address: `matsuki@tokyo-kasei.ac.jp`

I. INTRODUCTION

In the search for simplicity in the physics of color confinement, lattice gauge theory models have been studied extensively for a clue to a mechanism or an underlying principle governing the phenomenon. Spontaneous gauge symmetry breaking of the dual $U(1)$, and the resulting condensation of $U(1)$ monopole currents, defined after appropriate gauge fixing, remains a candidate. The persistent monopole currents of dual superconductivity in pure $U(1)$ lattice models leads to confinement of charge. There has been some success in the postulate of Abelian dominance in correlating monopoles and confinement physics but no breakthrough in uncovering a definitive mechanism. There are a number of reviews of the subject[1–9]. Some of the principal directions include accounting for string tension in Abelian Wilson loops[10, 11], similarly for monopole dominance of Wilson loops[12, 13], correlating percolating monopole clusters and confinement[14–17], spontaneous gauge symmetry breaking (SGSB) of dual $U(1)$ symmetry of the vacuum[18–20], and the subject of this paper the dual $U(1)$ Abrikosov vortex in the confining string[21–31].

Truncation to "relevant variables" invariably leads to systematic errors. Ambiguities due to gauge fixing[32–36] and Gribov copies[37] contribute to these perennial problems. Further it is difficult to see how this mechanism alone can explain all aspects of color confinement in arbitrary systems. Nevertheless one can argue that the well established phenomena can be part of a larger picture.

In this paper we are interested in testing spontaneous gauge symmetry breaking (SGSB) of dual $U(1)$ by examining the vortex in the confining string. We consider $SU(2)$ and, once fixed to the maximal Abelian gauge, we look at model independent results. High statistics data is readily available and there are a number of reasons to revisit the problem. It is possible to remove systematic errors due to lattice definitions in the divergence of the electric field which determines the total electric flux in the vortex. And then this can be used in a consistent definition of the magnetic current whose curl determines the profile of the solenoidal currents forming the vortex. This formulation of flux and currents differs from the more standard ones to order a^2 and higher order and presumably gives the same continuum limit. Finally we address a shortcoming of using the Ginzburg-Landau-Higgs (GLH) model to interpret the simulation data. All three effects are central to the determination of the type of dual superconductor; type I or II.

i) Exact electric Maxwell equations. There is a definition of flux that respects the *electric* Maxwell equations exactly. In 1998, DiCecio, Hart and Haymaker (DHH)[38], using the freedom of choosing non-leading terms in the lattice spacing a, derived a specific form that satisfies the $U(1)$ Ward-Takahashi identity giving exact electric Maxwell equations for lattice averages to all orders in lattice spacing a. Earlier, Zach, Faber, Kainz and Skala (ZFKS)[39] derived a similar relation for the $U(1)$ theory. With the DHH definition, the charge density induced in the neighborhood of the sources and the resulting electric flux in the string are free of systematic errors that are due to lattice definitions. We further find here that there are also significant electric currents that are important in the analysis (see point (iii) below). This is the first application of this formulation which we reported in proceedings in 2003[40].

ii) Consistent magnetic Maxwell equations. One can then also get a consistent exact *magnetic* Maxwell equation by adopting the same definition of flux in defining the magnetic current[40]. This is not the conventional procedure. The standard DeGrand-Toussaint (DT)[41] construction identifies cubes with quantized monopoles. By using the DHH flux instead, the current is conserved but is not quantized in cubes. Rather it is smeared out among neighboring cubes. In other problems the DT monopole construction is preferred. For example in describing percolation of clusters[14–17].

iii) GLH interpretation of simulation data. There are pitfalls in using the GLH model for a detailed fit to the simulation data. Most notably, a single parameter serves two roles. We use the notation Λ_d for the proportionality between the electric field and the curl of the magnetic current in the dual London relation which holds for sufficiently large transverse distance in the tail of the vortex for a type II dual superconductor. Secondly we use the notation $\lambda_d = 1/m_{\text{dual photon}}$ for the penetration depth of the electric field into a dual superconductor. Because of the presence of electric currents we show here that these two quantities take significantly different values in the $SU(2)$ theory though they take the same value in the GLH model. A fit of the simulation to the GLH model can at best be a compromise between these two quantities.

In Section II we present our method of determining model-independent estimates of Λ_d, λ_d, and ξ_d (= dual coherence length) $= 1/m_{\text{dual Higgs}}$. Appendix A reviews the GLH model to clarify our point of departure. (The results of Appendix A must be dual transformed when applying to the body of this paper.)

In Section III we review three definitions of flux. Appendices B and C give a review of the derivation of the two less familiar definitions. We make the argument for choosing the DHH form and discuss its normalization. We then argue that the DHH definition should be used to define magnetic currents rather than the standard DT construction.

The numerical results are given in Section IV. We show vortex profiles only for the case of Wilson loops of size $R = 7$ and $T = 3$ to show relevant behavior and then we give fitted parameter values for a wide range of R's and T's. Although the estimates have modest errors for small R and T, attempts to extract large R and T values remain problematic. Our best measurements are for the parameter λ_d which allows a stable extrapolation. The parameter Λ_d is poorly defined and shows excessive fluctuations. Nevertheless we make the case that the extrapolated value of λ_d lies outside all the error bars Λ_d for all measured values of R and T which supports our contention that the dual GLH model can not be applied directly in the interpreting of the simulation data.

Is this a type I or type II dual superconductor? As with all previous studies, we present evidence that $\lambda_d \approx \xi_d$ but we are not able to give a definitive result.

What is the effect of using the DT or ZFKS rather than the DHH definition of flux and magnetic current? We found a $10 \sim 30\%$ factor difference in their normalizations but no significant difference in the profiles of flux and current distributions. Depending on the choice this can lead to as much as 40% error in the determination of the dual superconductivity parameters.

As a digression from the main body of this paper, we show the effect of using DT definition of magnetic current compared to a truncated DT definition in which only the single dominant percolating cluster is included. The truncation of 60% of the monopoles making up the configuration has a few percent effect on the measurement of magnetic current in the presence of a fattened Wilson loop source.

Section V gives our summary and conclusions.

II. MODEL INDEPENDENT ANALYSIS OF THE DUAL VORTEX

The dual GLH model provides the inspiration for the analysis of the confining string in the $SU(2)$ theory in the Maximal Abelian gauge. However there are shortcomings in using this as a detailed model of the simulation data. There are electric currents in $SU(2)$ that have no counterpart in this model which, as we show, play a significant role in the analysis. Consequently, unlike the dual GLH model, the London penetration depth as determined by the London relation, Λ_d does not determine the exponential fall-off of the electric field profile of the vortex determined by λ_d. A fit of the simulation data to the dual GLH model requires a numerical solution of the coupled dual GLH non-linear Euler-Lagrange equations. The solution makes no distinction between these two occurences of the London penetration depth and hence a fit is compromised. The problems may or may not go away in the continuum limit but it definitely occurs in the simulation window of accessible values of β.

Nevertheless the dual Higgs model points the way to verify SGSB and estimate the London penetration depth and the coherence length without a detailed numerical solution.

A. Zero coherence length

The simplest circumstance is an extreme type II dual superconductor which can be modeled by assuming a constrained dual Higgs field

$$\Phi_d(\boldsymbol{m}) = v_d \rho_d(\boldsymbol{m}) e^{i\chi_d(\boldsymbol{m})}; \ \rho_d(\boldsymbol{m}) = \text{const.} = 1. \tag{1}$$

This leads immediately to a dual London relation satisfied everywhere (except on axis) in a vortex solution aligned in the z direction and independent of z and t.

$$\mathcal{E}_z \ = \ \Phi_E \delta^2(\boldsymbol{r}_\perp), \tag{2}$$

where the "fluxoid" is defined

$$\mathcal{E}_z \ \equiv \ E_z - \Lambda_d^2 \left(\text{curl} J^{(m)}\right)_z . \tag{3}$$

Both the profiles E_z and $\left(\text{curl} J^{(m)}\right)_z$ have decaying exponential behavior with a London penetration length $\lambda_d = 1/m_{\text{dual photon}}$. Cancelation of the profiles fixes the value of Λ_d. Integrating both sides of Eq.(2) on the transverse plane gives

$$\int d^2\boldsymbol{r}_\perp E_z \ = \ \Phi_E, \tag{4}$$

verifying that Φ_E is the total electric flux in the vortex. The vanishing of $\int d^2\boldsymbol{r}_\perp \left(\text{curl} J^{(m)}\right)_z$ follows from Stokes theorem and the fact that $\left(\text{curl} J^{(m)}\right)_z$ vanishes exponentially with $\boldsymbol{r}_\perp$.

B. Finite coherence length

The above example describes a discontinuous transition from a dual superconductor everywhere to the normal phase on axis where $\rho_d(\boldsymbol{m}) = 0$. However for the $SU(2)$ theory the transition is smooth and the delta function is replaced by a smooth function $f(\boldsymbol{r}_\perp)$ which decays over a distance scale $\xi_d = 1/m_{\text{dual Higgs}}$.

$$\mathcal{E}_z \;=\; \Phi_E f(\boldsymbol{r}_\perp); \quad \int d^2\boldsymbol{r}_\perp f(\boldsymbol{r}_\perp) = 1. \tag{5}$$

The tails of the profiles in Eq.(5) can still be used to determine Λ_d as long as there is a sufficiently large domain in $\boldsymbol{r}_\perp$ in which the asymptotic profiles match. Hence this analysis requires $\lambda_d > \xi_d$.

With the cancelation of the leading behavior in Eq.(5), ξ_d can be estimated from the behavior of $f(\boldsymbol{r}_\perp)$. The criterion for type II is

$$\kappa \equiv \frac{\lambda_d}{\xi_d} > \sqrt{\frac{1}{2}}. \tag{6}$$

C. Effects due to electric currents

Consider the magnetic Maxwell equations

$$J^{(m)}_\beta \;=\; -\frac{1}{2}\epsilon_{\beta\gamma\mu\nu}\,\frac{1}{a}\Delta^+_\gamma F_{\mu\nu}. \tag{7}$$

Take the curl of this

$$\begin{aligned}
-\epsilon_{\sigma\rho\alpha\beta}\Delta^-_\alpha J^{(m)}_\beta \;&=\; \left\{\epsilon_{\sigma\rho\alpha\beta}\frac{1}{2}\epsilon_{\beta\gamma\mu\nu}\right\}\frac{1}{a}\,\Delta^-_\alpha\Delta^+_\gamma F_{\mu\nu}, \\
&= \frac{1}{a}\left\{\Delta^-_\alpha\Delta^+_\sigma F_{\alpha\rho} - \Delta^-_\alpha\Delta^+_\rho F_{\alpha\sigma}\right. \\
&\qquad \left. + \Delta^-_\alpha\Delta^+_\alpha F_{\rho\sigma}\right\}.
\end{aligned} \tag{8}$$

Using the electric Maxwell equation,

$$\frac{1}{a}\Delta^-_\alpha F_{\rho\alpha} = J^{(e)}_\rho, \tag{9}$$

we find get the following relation

$$\epsilon_{\sigma\rho\alpha\beta}\Delta^-_\alpha J^{(m)}_\beta = \left(\Delta^+_\sigma J^{(e)}_\rho - \Delta^+_\rho J^{(e)}_\sigma\right) - \Delta^-_\alpha\frac{1}{a}\Delta^+_\alpha F_{\rho\sigma}. \tag{10}$$

Had we used different definitions of flux in the two currents Eqs.(7) and (9), then there would be violations to Eq.(10).

This relation is also satisfied by lattice averages *only if* one adopts the DHH definition of flux everywhere.

Applying this to a tail of a vortex oriented along the z axis and looking at $\left(\mathrm{curl}J^{(m)}\right)_z$

$$\begin{aligned}\frac{1}{a}\left(\Delta_1^- J_2^{(m)} - \Delta_2^- J_1^{(m)}\right) - \frac{1}{a^2}\left(\Delta_1^-\Delta_1^+ + \Delta_2^-\Delta_2^+\right)F_{34} \; = \\ \frac{1}{a}\left(\Delta_3^+ J_4^{(e)} - \Delta_4^+ J_3^{(e)}\right) + \frac{1}{a^2}\left(\Delta_3^-\Delta_3^+ + \Delta_4^-\Delta_4^+\right)F_{34}.\end{aligned} \tag{11}$$

The electric current can survive in the dual GLH model but only as a lattice artifact and it vanishes in the continuum limit. Further the second derivative terms on the RHS of Eq.(11) are designed to be as small as possible by the choice of the source. Assume first that the RHS vanishes. Then the London relation becomes

$$E_z - \Lambda_d^2\left(\mathrm{curl}J^{(m)}\right)_z = E_z - \lambda_d^2\nabla_\perp^2 E_z = 0; \qquad \Lambda_d^2 = \lambda_d^2.$$

This clearly identifies λ_d^2 as a penetration depth in the dual superconductor.

However the RHS does *not* vanish for lattice averages in our simulation. In the standard simulation window, the terms on the RHS are of the same order as the terms on the LHS. Hence the value Λ_d as measured by the London relation in the tail of the profile does *not* control the rate of transverse fall-off of the profile. In a dual GLH model it does. For this reason we choose not to rely on a fit to the dual GLH model but concentrate instead on verifying the model-independent SGSB, and estimating the three parameters Λ_d, λ_d, and ξ_d.

III. THREE DEFINITIONS OF FLUX

Let us consider three definitions of field strength or flux, all agreeing to lowest order in the lattice spacing a. If we require that both electric and magnetic Maxwell equations for lattice averages of flux and current be satisfied, then the specific form of the action implies a unique definition of flux. Using any of the alternative definitions introduces contributions, non-leading in a, that violate Maxwell's equations. Consistency would be restored only by going to the continuum limit. We see no advantage in introducing such unnecessary complications and argue for exact consistency.

A. DeGrand-Toussaint[41](DT): $\widehat{F}_{\mu\nu}^{(1)}$

The first definition is that used by DeGrand and Toussaint to define monopoles in the $U(1)$ theory:

$$\begin{aligned}\widehat{F}_{\mu\nu}^{(1)}(\boldsymbol{n}) &\equiv \theta_{\mu\nu}(\boldsymbol{n}) - 2\pi n_{\mu\nu}(\boldsymbol{n}), \\ \theta_{\mu\nu}(\boldsymbol{n}) &\equiv \theta_\mu(\boldsymbol{n}) - \theta_\mu(\boldsymbol{n}+\nu) - \theta_\nu(\boldsymbol{n}) + \theta_\nu(\boldsymbol{n}+\mu),\end{aligned} \tag{12}$$

where θ_μ refers to the $U(1)$ link angle in the domain $-\pi < \theta_\mu < +\pi$. The integers $n_{\mu\nu}$ are determined by requiring that $-\pi < \widehat{F}_{\mu\nu}^{(1)} < +\pi$. That is $\widehat{F}_{\mu\nu}^{(1)}$ is a periodic function of $\theta_{\mu\nu}$ with period 2π (Quantities with $\widehat{}$ mean those which appear in the lattice simulation without appending factors of the gauge coupling constant e or g and lattice spacing a.) We also refer to $\widehat{F}_{\mu\nu}^{(1)}$ as the 'sawtooth' flux as shown in Fig.1.

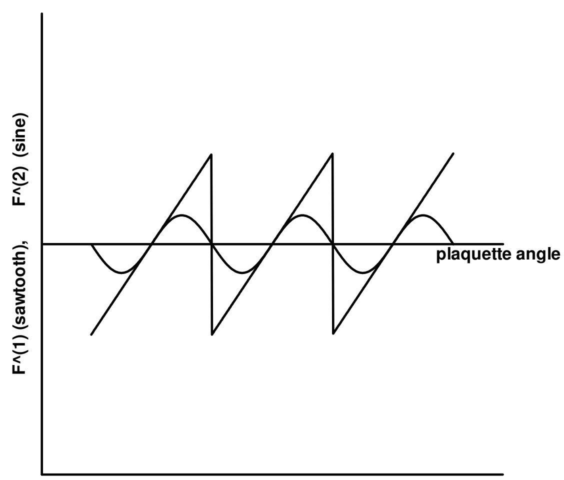

FIG. 1: $\widehat{F}_{\mu\nu}^{(1)}$ (sawtooth) and $\widehat{F}_{\mu\nu}^{(2)}$ (sine) as a function of the plaquette angle $\theta_{\mu\nu}$

.

B. Zach-Faber-Kainz-Skala[39](ZFKS): $\widehat{F}_{\mu\nu}^{(2)}$

A second definition

$$\widehat{F}_{\mu\nu}^{(2)}(\boldsymbol{n}) \equiv \sin\theta_{\mu\nu}(\boldsymbol{n}), \tag{13}$$

has the property of giving the exact *electric* Maxwell equation for lattice averages for the case of the $U(1)$ gauge theory with Wilson action

$$\beta\Delta_\nu^- \left\langle \widehat{F}^{(2)}_{\mu\nu} \right\rangle_W = \widehat{J}^{(e)}_\mu, \tag{14}$$

where

$$\langle\cdots\rangle_W = \frac{\langle \sin\theta_W \cdots\rangle}{\langle\cos\theta_W\rangle}.$$

The current $\widehat{J}^{(e)}_\mu$ is that carried by the Wilson loop, normalized to take values $-1, 0, 1$. We review the derivation in Appendix B.

C. DeCecio-Hart-Haymaker[38](DHH): $\widehat{F}^{(3)}_{\mu\nu}$

And finally a third definition, applicable specifically to Abelian projected $SU(2)$,

$$\begin{aligned}\widehat{F}^{(3)}_{\mu\nu}(\boldsymbol{n}) \equiv\; & C_\mu(\boldsymbol{n})C_\nu(\boldsymbol{n}+\mu)C_\mu(\boldsymbol{n}+\nu)C_\nu(\boldsymbol{n}) \times \\ & \sin\theta_{\mu\nu}(\boldsymbol{n}),\end{aligned} \tag{15}$$

where the link variables are parameterized by $\theta_\mu(\boldsymbol{n})$, $\phi_\mu(\boldsymbol{n})$ and $\gamma_\mu(\boldsymbol{n})$

$$U_\mu(\boldsymbol{n}) = \begin{pmatrix} C_\mu e^{i\theta_\mu} & S_\mu e^{i(\gamma_\mu-\theta_\mu)} \\ -S_\mu e^{-i(\gamma_\mu-\theta_\mu)} & C_\mu e^{-i\theta_\mu} \end{pmatrix}, \tag{16}$$

and where

$$\begin{aligned} C_\mu(\boldsymbol{n}) &\equiv \cos\phi_\mu(\boldsymbol{n}), \\ S_\mu(\boldsymbol{n}) &\equiv \sin\phi_\mu(\boldsymbol{n}). \end{aligned}$$

This form has the property of giving the exact *electric* Maxwell equation for lattice averages for this case of $SU(2)$ in the maximal Abelian gauge with Wilson action

$$\beta\Delta_\nu^- \left\langle \widehat{F}^{(3)}_{\mu\nu} \right\rangle_{W,g.f.} = \left\langle \widehat{J}^{(e)\mathrm{total}}_\mu \right\rangle_{W,g.f.}. \tag{17}$$

Where $\widehat{J}^{(\mathrm{e})\ \mathrm{total}}_\mu$ gets contributions from the Abelian Wilson loop, the charged matter fields, gauge fixing and ghosts. The expectation value of the current

$$\left\langle \widehat{J}^{(e)\mathrm{total}}_\mu \right\rangle_{W,g.f.}$$

includes the contribution from the Wilson loop $\widehat{J}^{(e)}_\mu$ normalized the same as in the ZFKS case in the previous section.

In Appendix C we sketch the derivation of this result.

D. Consistency with the magnetic Maxwell equation

For the second and third cases we have a unique flux $\widehat{F}^{(i)}_{\mu\nu}$, for $i = 2, 3$, by requiring an exact lattice *electric* Maxwell equation. Given this definition of flux the *magnetic* Maxwell equation is

$$-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Delta^{+}_{\nu}\widehat{F}^{(i)}_{\rho\sigma} \;=\; \widehat{J}^{(m)}_{\mu} \qquad i = 2, 3.$$

which gives a unique definition of the magnetic current. However the monopole current is usually taken from the DT definition

$$\widehat{J}^{(m)}_{\mu} \;=\; -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Delta^{+}_{\nu}\widehat{F}^{(1)}_{\rho\sigma}.$$

(This current is normalized to give monopoles with a flux of $2\pi n$ where n is integer.) Hence if we use the conventional $\widehat{F}^{(1)}$ to define the monopole current, and $\widehat{F}^{(2)}$ or $\widehat{F}^{(3)}$ respectively for $U(1)$ and $SU(2)$ theories to get an exact expression for flux in the confining string, then the magnetic Maxwell equation is violated.

The *electric* Maxwell equation determines the total electric flux in the confining string and the *magnetic* Maxwell equation determines the transverse profile through the solenoidal currents. The only way for the calculation to be consistent with both Maxwell equations is to relax the usual procedure using the DT monopole definition and instead use $\widehat{F}^{(2)}$ or $\widehat{F}^{(3)}$ when defining magnetic currents for the $U(1)$ and $SU(2)$ cases, respectively.

A simple configuration will help illustrate the difference between $\widehat{F}^{(1)}$ and $\widehat{F}^{(2)}$. Consider a single DT monopole with equal flux out of the six faces of the cube (and a Dirac string extending out from any face). Then the ratio of the $\widehat{F}^{(2)}$ flux out of this cube compared to the $\widehat{F}^{(1)}$ flux gives

$$\frac{6\sin(2\pi/6)}{6(2\pi/6)} \approx 0.83. \tag{18}$$

On a large surface the total flux is the same for the two definitions. Since charge is conserved, the balance is made up by magnetic charge in the neighboring cubes. We interpret this to mean that with magnetic currents defined with $\widehat{F}^{(2)}$, the discrete monopoles become smeared but maintain the same total magnetic charge.

E. Comparison

Figure 1 shows a comparison between the first two definitions. We plot $\widehat{F}^{(1)}_{\mu\nu}$ as a function of $\theta_{\mu\nu}$, giving a "sawtooth" shape. Monopoles occur as a consequence of $\theta_{\mu\nu}$ crossing the sawtooth edge, giving a mismatch of 2π in the flux out of a cube. The sine function, $\widehat{F}^{(2)}_{\mu\nu}$, has no such discreteness and so the notion of discrete Dirac strings and Dirac monopoles is absent. However as one approaches the continuum limit, the action will drive the plaquette angle to zero, mod 2π, and then the regions where the sawtooth differs from the sine function are suppressed. Hence we expect both forms to give the standard Dirac picture in the continuum limit.

The $\widehat{F}^{(3)}_{\mu\nu}$ definition is a modification of $\widehat{F}^{(2)}_{\mu\nu}$ involving factors of C_μ, the cosine of the matter fields in the Abelian projection of the $SU(2)$ variables. Since $C_\mu \approx 1 + O(a^2)$ the fluctuations of this factor are suppressed in the maximal Abelian gauge as noted by Poulis[42]. In the simulations described here, our measurements of field strength have errors at best of the order of 1.0% whereas the error in $\langle C_\mu \rangle$ is 0.01%.

F. Physical normalization of the field strengths

The physical dimension of the fields is obtained from the leading order of a small a limit. The definition of electric charge in the $U(1)$ theory is straightforward and well known. The Wilson loop carries the charge of the gauge coupling constant.

The Abelian projected $SU(2)$ case is not as clear cut. Measurement of flux at the site of the Wilson loop includes the bare charge and dynamical contributions screening attributed to dynamical charged matter fields, gauge fixing and ghosts. Further the matter fields can not be completely disentangled from the gauge fields. We propose a normalization that involves the effects of these matter fields.

1. $U(1)$ theory with Wilson Action

As derived in Appendix B, we have the exact electric Maxwell equation

$$\beta \Delta^-_\nu \left\langle \widehat{F}^{(2)}_{\mu\nu} \right\rangle_W = \widehat{J}^{(e)}_\mu . \tag{19}$$

With the Wilson action

$$S = \sum_{n,\, \mu>\nu} \left(\cos\theta_{\mu\nu}(\boldsymbol{n}) - 1\right), \tag{20}$$

one arrives in the usual way that

$$\beta = \frac{1}{e^2}.$$

Rewriting Eq.(19) to display the physical normalization of the field strength gives

$$\begin{aligned} \frac{\Delta_\nu^-}{a}\left\langle \frac{\widehat{F}^{(2)}_{\mu\nu}}{ea^2} \right\rangle_W &= e\frac{\widehat{J}^{(e)}_\mu}{a^3}, \\ \frac{\Delta_\nu^-}{a}\left\langle F^{(2)}_{\mu\nu}\right\rangle_W &= J^{(e)}_\mu. \end{aligned}$$

The current $\widehat{J}^{(e)}_\mu$ is normalized to unity on the Wilson loop, (see Appendix B).

Consider the magnetic Maxwell equation where we advocate the consistent use of $\widehat{F}^{(2)}_{\mu\nu}$ giving

$$-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Delta_\nu^+\widehat{F}^{(2)}_{\rho\sigma} \;=\; \widehat{J}^{(m)}_\mu,$$

In this case

$$\begin{aligned} -\frac{\Delta_\nu^+}{a}\left(\frac{\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\widehat{F}^{(2)}_{\rho\sigma}}{ea^2}\right) &= \frac{1}{e}\frac{\widehat{J}^{(m)}_\mu}{a^3}, \\ -\frac{\Delta_\nu^+}{a}\left(\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{(2)}_{\rho\sigma}\right) &= J^{(m)}_\mu. \end{aligned}$$

The magnetic charge is

$$e_m \;=\; \frac{2\pi}{e}.$$

Recall that $\widehat{J}^{(m)}_\mu$ is normalized to take the value 2π for a DT monopole. The argument in subsection D above shows that the same normalization holds for a smeared monopole constructed from ZFKS flux.

In summary

$$F^{(2)}_{\mu\nu} \;=\; \frac{1}{ea^2}\widehat{F}^{(2)}_{\mu\nu}, \tag{21}$$

$$J^{(e)}_\mu \;\equiv\; \frac{e}{a^3}\widehat{J}^{(e)}_\mu, \tag{22}$$

$$J^{(m)}_\mu \;\equiv\; \frac{2\pi}{e}\left(\frac{1}{2\pi a^3}\widehat{J}^{(m)}_\mu\right). \tag{23}$$

Eq.(C5) in Appendix C gives the analogous starting point for this case.

$$\begin{aligned}\beta\Delta_\nu^- \left\langle \widehat{F}^{(3)}_{\mu\nu} \right\rangle_{W,g.f.} &= \widehat{J}^{(e)\text{Abelian Wilson loop}}_\mu \\ &\quad + \left\langle \widehat{K}^{(e)}_\mu \right\rangle_{W,g.f.},\end{aligned} \tag{24}$$

where $\widehat{K}^{(e)}_\mu$ is the sum of the three dynamical terms in the current. We see from Eq.(C4) that as in the $U(1)$ case $\widehat{J}^{(e)\text{Abelian Wilson loop}}_\mu$ is also normalized to unity on the Wilson loop.

Consider the action

$$\beta \sum_P \frac{1}{2}\mathrm{Tr}(U_{\mu\nu}) = \beta \sum_P \frac{1}{2}\mathrm{Tr}(D_{\mu\nu}) + \beta\widetilde{S},$$

where $D_{\mu\nu}$ is the plaquette formed from the diagonal part of the each of the four links U_μ and $\widetilde{S}$ is the remainder involving couplings of the matter fields among themselves and to the $U(1)$ gauge field. Further, write

$$\begin{aligned}\beta \sum_P \frac{1}{2}\mathrm{Tr}(D_{\mu\nu}) = \beta \sum_P C_\mu(\boldsymbol{m})C_\nu(\boldsymbol{m}+\mu)\times \\ C_\mu(\boldsymbol{m}+\nu)C_\nu(\boldsymbol{m})\left\{\cos\theta_{\mu\nu}(\boldsymbol{m})-1\right\} + \beta\widetilde{\widetilde{S}},\end{aligned}$$

where $\widetilde{\widetilde{S}}$ compensates for the subtraction in the curly bracket which involves only charge neutral combinations of the matter fields. For the maximal Abelian gauge we will make use of the fact that the fluctuations of C_μ are suppressed. If we take C_μ to be constant and compare with the $U(1)$ action, Eq.(20) we can introduce the $U(1)$ charge through the relation.

$$\frac{1}{e^2} = \beta\left\langle C_\mu\right\rangle^4 . \tag{25}$$

Then we can write

$$\begin{aligned}\frac{\Delta_\nu^-}{a}\left\langle \frac{\widehat{F}^{(3)}_{\mu\nu}}{ea^2\left\langle C_\mu\right\rangle^4}\right\rangle_{W,g.f.} &= e\frac{\widehat{J}^{(e)\text{Abelian Wilson loop}}_\mu}{a^3} \\ &\quad + \left\langle e\frac{\widehat{K}^{(e)}_\mu}{a^3}\right\rangle_{W,g.f.}.\end{aligned}$$

The novel aspect of this definition of flux is the existence of the C_μ factors. These are required to get an exact WT identity. The suppression of the flux by these factors is partially compensated by the presence of $\langle C_\mu \rangle^4$ in the normalization. In the simulation for $\beta = 2.5115$ we found $\langle C_\mu \rangle = 0.94784(4)$ which gives

$$e = 0.7024(1). \tag{26}$$

This is the charge carried by the bare Abelian Wilson loop. Equation (26) is only an estimate of the value of e because $\langle C_\mu \rangle$ is not strictly a constant.

It is beyond the scope of this paper to make general statements about the magnetic current for this case. For $\widehat{F}^{(1)}_{\mu\nu}$ and $\widehat{F}^{(2)}_{\mu\nu}$, the magnetic current is defined through the violation of the Bianchi identity and unlike the electric current does not involve expectation values. For the purpose of this calculation and the observation that C_μ appears to be essentially constant over the profile of the vortex and has small fluctuations, we get a physical normalization for the magnetic current analogous to the $U(1)$ case. However this need not be the correct approach for a discussion of the charge of static monopole for example since it must involve assumptions about the behavior of the matter field ϕ_μ. Our approach here is to introduce an expectation value of C_μ and hence this is not as general as the other cases. This normalization does not impact our determination of the dual superconductivity parameters since the same factors appear in the two terms of the London relation.

Consider the magnetic Maxwell equation where we advocate the consistent use of $\widehat{F}^{(3)}_{\mu\nu}$ giving

$$-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Delta^+_\nu \widehat{F}^{(3)}_{\rho\sigma} \;=\; \widehat{J}^{(m)}_\mu,$$

In this case

$$\begin{aligned}
-\frac{\Delta^+_\nu}{a}\left(\frac{\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\widehat{F}^{(3)}_{\rho\sigma}}{ea^2\left\langle C_\mu\right\rangle^4}\right) &= \frac{1}{e}\frac{\widehat{J}^{(m)}_\mu}{a^3\left\langle C_\mu\right\rangle^4}, \\
-\frac{\Delta^+_\nu}{a}\left(\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{(3)}_{\rho\sigma}\right) &= J^{(m)}_\mu.
\end{aligned}$$

In summary

$$F^{(3)}_{\mu\nu} = \frac{1}{ea^2\left\langle C_\mu\right\rangle^4}\widehat{F}^{(3)}_{\mu\nu}, \tag{27}$$

$$J^{(e)}_\mu = \frac{e}{a^3}\widehat{J}^{(e)}_\mu, \tag{28}$$

$$J^{(m)}_\mu = \frac{1}{ea^3\left\langle C_\mu\right\rangle^4}\widehat{J}^{(m)}_\mu. \tag{29}$$

IV. SIMULATION

Our measurements were on 208 gauge-fixed configurations on a 32^4 lattice, with $\beta = 2.5115$. Each update consisted of a 10 hit metropolis sweep and an overrelaxation sweep. We made 13 runs on 16 independent nodes. Dropping 2000 thermalization updates (on each node), we made measurements on every $100th$ update.

We gauge-fixed to the Maximal Abelian Gauge (MAG) using overrelaxation with the criterion of the average of the absolute value of the off-diagonal matrix element of the MAG adjoint operator $< 10^{-6}$.

We measured Wilson loops in which spacial links were fattened through 100 iterative steps by adding spacial staples of weight equal to the original link. We found no measurable sensitivity to the weighting factor or to increasing the number of iterations and hence the fattening is saturated.

All vortex profile graphs presented here are transverse slices through the mid-plane, the (x, y) plane, on the quark-antiquark axis, the z axis. *We chose fattened Wilson loop of size $R/a = 7$ and $T/a = 3$ in all profile graphs to show relevant behavior.* Fitted parameter values are given for 6 quark-antiquark separations $R/a = 3, 5, \ldots, 13$ and for 7 time separation of $T/a = 3, 4, \ldots, 9$. All axes in the graphs are dimensionless.

We used $a^2 F^{(3)}_{\mu\nu}$ throughout for the definition of flux in order to give the correct electric Maxwell equations. Similarly we used the same definition in constructing the magnetic current $a^3 J^{(m)}_{\mu}$ in order to get the correct magnetic Maxwell equation. (Exceptions of course include graphs comparing different definitions and truncated vs. complete monopole loops.)

Noise in these data is a problem. We reduced this by taking an azimuthal angular average over an annular region of width = 1 in lattice units in the transverse, (x, y) plane, weighting the data by the fractional area overlap of the data point plaquette to the annulus of radius r

$$\langle \cdots \rangle_\phi \; = \; \frac{1}{2\pi r} \int_{r<r'<(r+1)} (\cdots) da'. \tag{30}$$

Using this together with Stokes theorem we have a convenient numerical evaluation of

$$
\begin{aligned}
\oint_{r'=r} \boldsymbol{J} \cdot d\boldsymbol{r'} &= 2\pi r \left\langle J_\phi(r) \right\rangle_\phi , \\
&= \int_{r'<r} \mathrm{curl}\boldsymbol{J} \cdot d\boldsymbol{a'}, \\
&= - \int_{r<r'} \mathrm{curl}\boldsymbol{J} \cdot d\boldsymbol{a'},
\end{aligned}
\tag{31}
$$

where we used the identity $\int \mathrm{curl}\boldsymbol{J} \cdot d\boldsymbol{a'} = 0$.

Since the current is a first derivative of the flux and the curl of the current is a second derivative, errors become more difficult for the latter. However by integrating an equation over a transverse area involving the curl we are back to first derivatives which are more manageable.

Except where a fit to the data is noted, the lines in the graphs connect the data points. All axes on all graphs are dimensionless. For physical normalizations, one can take the standard value $a = 0.086\mathrm{fm} = 0.44 GeV^{-1}$.

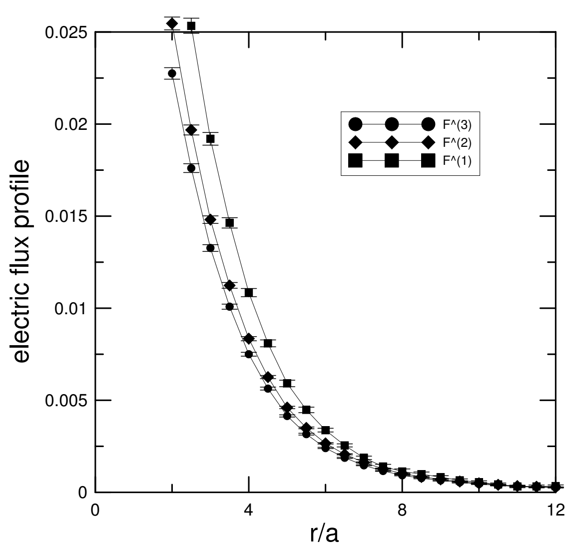

FIG. 2: Profiles for three definitions of the electric flux $a^2 F_{34}^{(1,2,3)}$ through the transverse plane

A. Comparing flux definitions

Recall the three definitions of flux: $F_{\mu\nu}^{(1,2,3)}$: Eq.(12, 13, 15). Figure 2 shows the E_z profile for the three definitions of flux. For a large variety of quark separations $R = 3, 5, \ldots, 13$

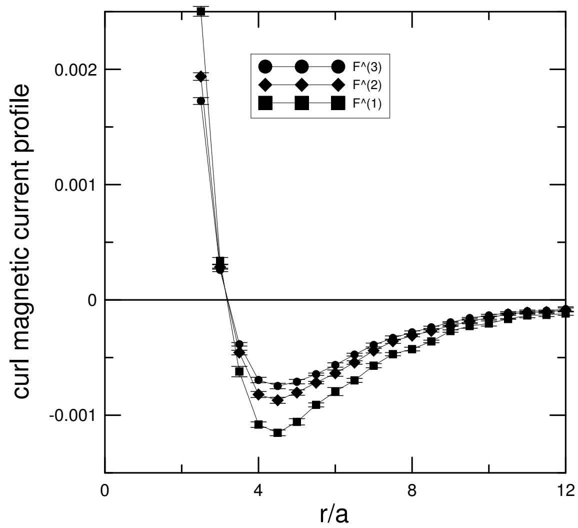

FIG. 3: Profiles for the z component of $-a^3 \mathrm{curl} J^{(m)}$ based on three definitions of the electric flux $a^2 F_{\mu\nu}^{(1,2,3)}$.

and time extents $T = 3, 4, \ldots, 9$ the three definitions differ only in the scale. There is no significant difference in shape. The scale factors over this range are approximately constant.

$$\begin{aligned} \left\langle F_{\mu\nu}^{(1)} \right\rangle_W &\approx 1.3 \left\langle F_{\mu\nu}^{(2)} \right\rangle_W , \\ \left\langle F_{\mu\nu}^{(2)} \right\rangle_W &\approx 1.1 \left\langle F_{\mu\nu}^{(3)} \right\rangle_W . \end{aligned} \tag{32}$$

We note that if the small fluctuations were in fact absent in C_μ, Eq.(15), then $F_{\mu\nu}^{(3)} = F_{\mu\nu}^{(2)}$ since the C_μ factors cancel out in the normalization, Eq.(27).

Figure 3 shows the z component of curl $J^{(m)}$ constructed from the three definitions of flux. The same observations hold here as for the flux. Note that the connecting lines cross close to zero, further indicating just a scale factor. They scale with the same factors as the flux to about 1%.

Figures 2 and 3 show qualitatively the signal for a dual Abrikosov vortex no matter which definition is chosen. For large transverse distance, in this case $r/a > 5$, the London relation holds, i.e. the tails can be arranged to cancel,

$$\text{fluxoid} : \mathcal{E}_z \equiv E_z - \Lambda_d^2 \left(\mathrm{curl} J^{(m)} \right)_z \approx 0. \tag{33}$$

The integral of the $(\mathrm{curl} J^{(m)})_z$ over the plane vanishes hence the integrand must change sign. For an extreme type II dual superconductor, the profiles match and cancel everywhere except for a delta function contribution at $r = 0$. In our case the scale defining the breakdown of the London relation is defined to be the coherence length.

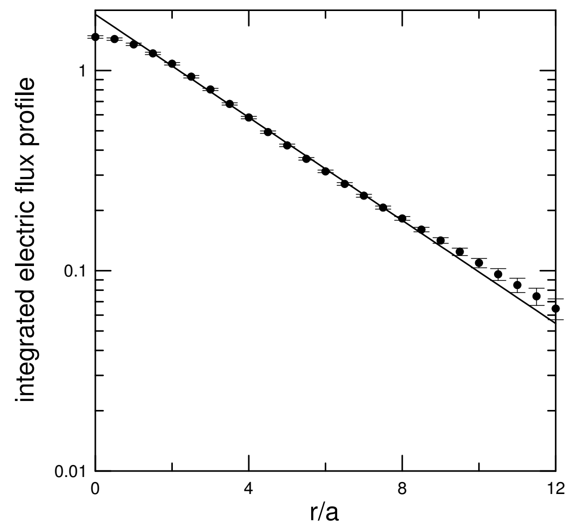

FIG. 4: Exponential fit to the electric flux profile

.

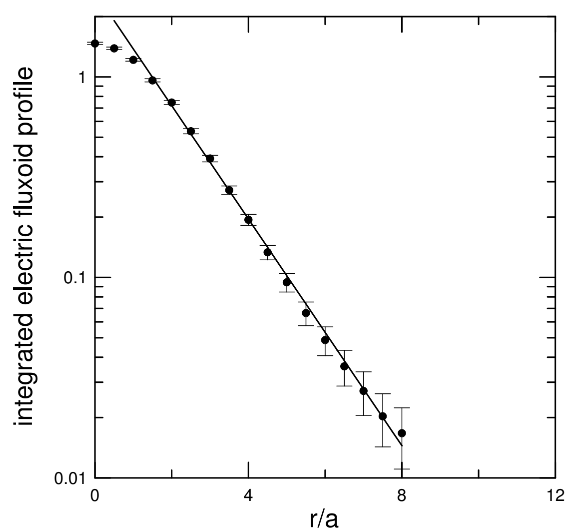

FIG. 5: Exponential fit to the electric fluxoid profile

.

B. Three parameters of the vortex

Figure 4 shows the integrated flux profile with a fit to an exponential

$$\int_{r'>r} a^2 E_z \ da' \ = \ Ae^{-r/\lambda_d}$$

As explained above the choice of integrating reduces noise in the signal.

For a very long vortex, it becomes a two dimensional problem and one would expect a profile for large r to be $K_0(mr) \sim e^{-mr}/\sqrt{r}$. For the integrated flux profile $\sim \int \sqrt{r}e^{-mr}dr$. With or without integration, neither gives a pure exponential behavior. But further the range of possible quark separations R/a does not put us in the domain of a pure two-dimensional problem. Without trying to correct for these effects by doing fits with more parameters, we accept the simple exponential form as an *estimate* of the parameters based on the a posteriori quality of the fits. We did a χ^2 fit at 23 points for $r/a = 1, 1.5, 2, \dots 12$. Except for a few cases at small R/a and small T/a, $\chi^2/d.f. < 1.0$ in the fits for all cases.

Next we fit in a similar fashion

$$\int_{r'>r} \left\{ a^2 E_z - \frac{\Lambda_d^2}{a^2}\left(a^4 \mathrm{curl} J^{(m)}\right)_z \right\} \ da' \ = \ Be^{-r/\xi_d}$$

determining Λ_d, B, and ξ_d. Figure 5 shows the sub-leading behavior after canceling the leading large r behavior between E_z and $(\mathrm{curl} J^{(m)})_z$, i.e. the fluxoid profile. Also shown is the fit.

Figures 6 and 7 shows the fitted values of Λ_d/a, λ_d/a and ξ_d/a. We used Minuit and quote Minuit errors in the parameters determined by $\Delta\chi^2 = 1$. Each group corresponds to the seven values of $T/a = 3, 4, \dots, 9$. Despite many efforts to reduce fluctuations, the errors grow rapidly with increasing R/a and T/a. Nevertheless Fig. 8, shows at attempt to extract the large R and T limits of some of this data with moderate success which we explain below. But let us first observe that Λ_d is poorly defined and the scatter of values in Fig. 6 defies such analysis. At best we speculate that Λ_d/a takes a value ≈ 1 with a large error.

First some qualitative observations: In Fig. 7 λ_d and ξ_d are well separated for $T/a = 3$ for all values of R/a. Recall the criterion for type II dual superconductivity is given by Eq. (6). Therefore for these values of R and T, the system is behaving like type II. As one decreases the contamination of excited states, i.e. T/a increasing, the difference between these two quantities decreases.

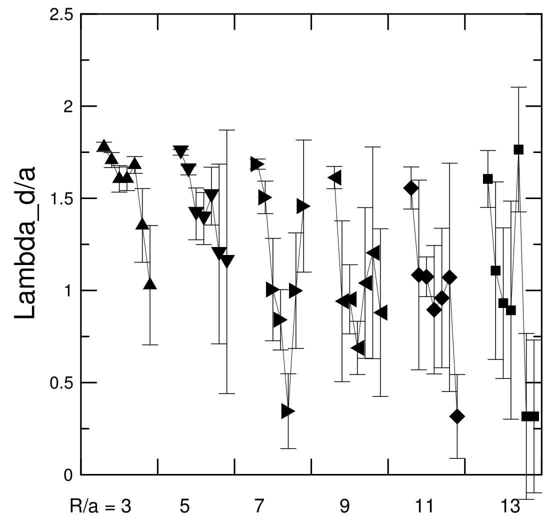

FIG. 6: Fitted values of Λ_d/a. Each group corresponds to $T/a = 3, 4, \ldots, 9$.

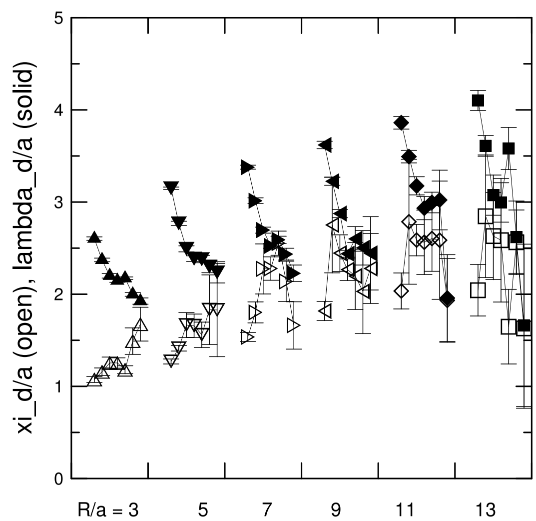

FIG. 7: Fitted values of λ_d/a (solid) and ξ_d/a (open). Each group corresponds to $T/a = 3, 4, \ldots, 9$.

The two quantities approach each other as T/a increases, i.e. for less contamination with excited states. Hence a system that is firmly type II is approaching type I in this limit.

A word of caution: if the condition $\xi_d > \lambda_d$ holds for the true solution, then the present analysis is not valid since we assume that a London relation exists sufficiently far out on the tail of the vortex. If further the true solution is type I then the approach in this paper is irrelevant since the dual GLH equations become unstable. Boundaries between superconducting and normal domains become energetically favored and serpentine surfaces develop.

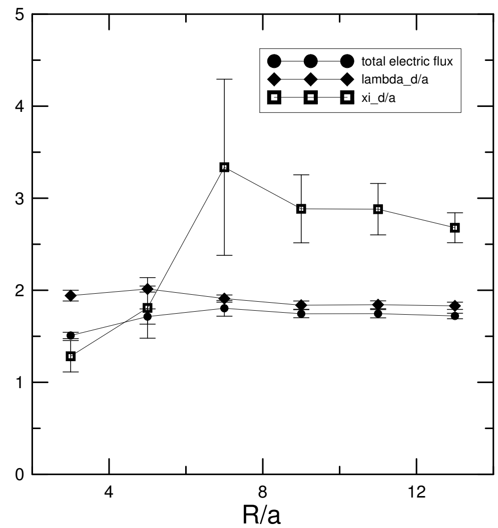

FIG. 8: A $R = \infty$ $T = \infty$ extrapolation of λ_d ξ_d, (Fig.7) and the integrated transverse flux (Fig. 9) based on data for spacial separations $3, 5, \ldots R/a$. The right-most points use all 42 data sets.

To get the large R and T limit, consider first the best case, i.e. λ_d in Fig.7. For $R/a = 3, 5$ before fluctuations obscure the data, an exponential fit appears to be reasonable

$$\lambda_d \; = \; A + Be^{-CT/a}$$

where A is the desired asymptote. In order to further extrapolate λ_d to large R we do a global fit modifying this by allowing an extra dependence linear in R/a in the parameters B and C to give

$$\lambda_d \; = \; A + (B + B' R/a)e^{-(C+C' R/a)T/a}$$

In Fig. 8 we include all points from $R/a = 3$ progressively up to $R/a = 3, 5, ...13$ giving a large T, large R extrapolated value

$$\lambda_d \; = \; 1.83(4).$$

A similar analysis for ξ_d gives

$$\xi_d \; = \; 2.68(16).$$

This last value should be approached with skepticism. The analysis requires $\lambda_d > \xi_d$ but this inequality is *reversed* for the extrapolated values. The χ^2 fits are heavily weighted to low values of R/a and T/a where the errors are small. Fig. 7 was included in this paper in order to provide an "eyeball" test to see if these numbers are reasonable. Since ξ_d involves the second derivative of the flux, it is inherently less reliable than λ_d. The safer conclusion is that these parameters tend to approach each other in the large R, large T limit. We regard it as fortuitous that we find a κ values, Eq.(6), close to the critical value, $1/\sqrt{2} = 0.707$, determining the boundary between a type I. vs type II dual superconductor,

$$\kappa \equiv \frac{\lambda_d}{\xi_d} = 0.68(4)$$

C. Total flux

In Fig. 9 we plot the integrated flux and fluxoid in the interior of the circle $r' \le r$

$$\int_{r' \le r} E_z \;\; da'; \qquad \int_{r' \le r} \mathcal{E}_z \;\; da'.$$

Both forms asymptote to the value of the total flux in the vortex but the fluxoid reaches the asymptote at a smaller value of r/a, as expected. Figure 10 shows the results as a function of R/a and T/a. We performed the same analysis as for Figs. 7 to obtain the large R/a

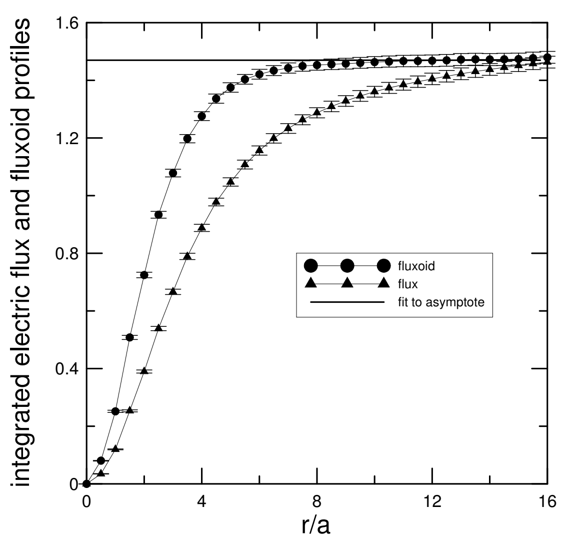

FIG. 9: integrated flux, integrated fluxoid

T/a limit. The results are included in Fig. 8.

$$\Phi_e = \int E_z \ da \ = \ 1.72(3)$$

Table I summarizes the relationship between the gauge coupling constant and the quantized vortex flux for the four cases considered here. For the $U(1)$ and $SU(2)$ cases, the Wilson loop carries the charge $e_{U(1)}$ and $e_{SU(2)}$ respectively. For the $U(1)$ case, the quantized unit of flux is also $e_{U(1)}$ and so the elementary charge produces exactly one unit of quantized flux in the vortex. In the $SU(2)$ case however there is a dynamical charge distribution generated by the source exhibited by the large value of $e_{\text{dynamical}} \equiv \Phi_{SU(2)} - e_{SU(2)} = 1.02(3)$. (Since Gauss' law is satisfied exactly in this formalism, $\Phi_{SU(2)}$ measures exactly the total charge on each side of the transverse plane.)

If we are correct that the proper interpretation of the simulation data implies that there is a dual gauge theory operating, fixing the quantization of flux to one unit, then from lines 2 and 4 in Table I we conclude that the fundamental unit of flux in that dual theory is $\Phi = 1.72(3)$, implying that the gauge coupling constant in that dual theory is $e_m = 2\pi/\Phi$.

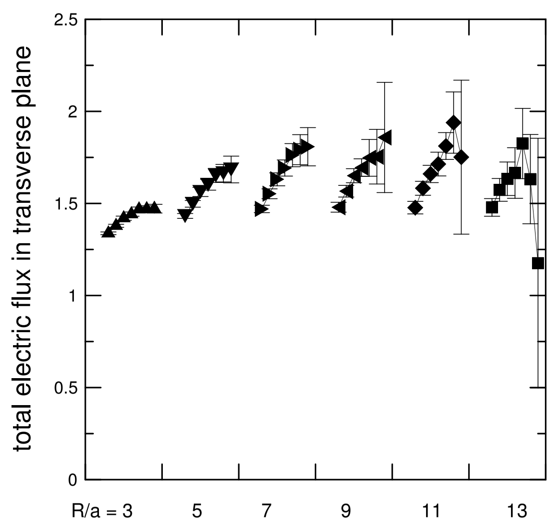

FIG. 10: Total electric flux in mid transverse plane. Each group corresponds to $T/a = 3, 4, \ldots, 9$.

D. Electric currents

Figure 11 shows an example of the effect of electric currents in this problem. The four graphs correspond to the four terms in the identity (a rewriting of Eq.(11) in dimensionless

	gauge action summand	gauge coupling	quantized vortex flux unit
GLH model	$\beta \cos\theta_{\mu\nu}$	$e = \beta^{-1/2}$	$\Phi_m = e_m = 2\pi/e$
dual GLH model	$\beta_{(d)} \cos\theta^{(d)}_{\mu\nu}$	$e_m = \beta_{(d)}^{-1/2}$	$\Phi_e = e = 2\pi/e_m$
$U(1)$ theory	$\beta_{U(1)} \cos\theta_{\mu\nu}$	$e_{U(1)} = \beta_{U(1)}^{-1/2}$	$\Phi_{U(1)} = e_{U(1)}$
$SU(2)$ MAG theory	$\approx \beta_{SU(2)} \langle C_\mu \rangle^4 \cos\theta_{\mu\nu} + \cdots$	$e_{SU(2)} = \beta_{SU(2)}^{-1/2} \langle C_\mu \rangle^{-2}$	$\Phi_{SU(2)} = e_{SU(2)} +$ (anti)screening
	$\langle C_\mu \rangle = 0.94784(4)$		
	$\beta_{SU(2)} = 2.5115$	$e_{SU(2)} = 0.7024(1)$	$\Phi_{SU(2)} = 1.72(3)$

TABLE I: Relationship between gauge coupling and quantized flux in the vortex.

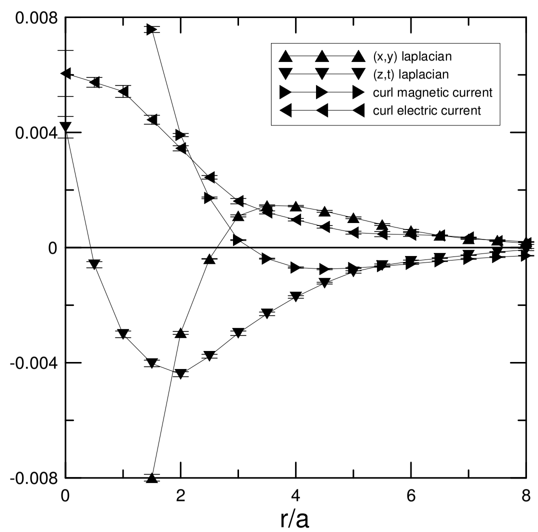

FIG. 11: The four contributions to the identity, Eq.(34). The sum of the four graphs vanishes identically.

form):

$$
\begin{aligned}
0 = a^3 \left(\Delta_1^- J_2^{(m)} - \Delta_2^- J_1^{(m)} \right) - a^2 \left(\Delta_1^- \Delta_1^+ + \Delta_2^- \Delta_2^+ \right) F_{34} \\
-a^3 \left(\Delta_3^+ J_4^{(e)} - \Delta_4^+ J_3^{(e)} \right) - a^2 \left(\Delta_3^- \Delta_3^+ + \Delta_4^- \Delta_4^+ \right) F_{34}.
\end{aligned} \quad (34)
$$

The curl of the magnetic current is in the (x, y) plane and the curl of the electric current is in the (z, t) plane. This electric contribution is in addition to the non-vanishing electric charge density in the neighborhood of the sources as noted by Bali et. al.[25].

In a continuum dual GLH model, the third term, the electric current term, vanishes.

In the lattice version, this current can appear but only as a lattice artifact. (See the GLH model, Appendix A and take the dual.) Further if we consider an infinitely long static vortex in the dual GLH model, then the fourth term also vanishes.

Both the third and fourth terms are significant and remain so over a wide range of R's and T's presented here. The discrepancy between Λ_d and λ_d further support the presence the third and fourth terms over this range.

E. Truncated monopole loops

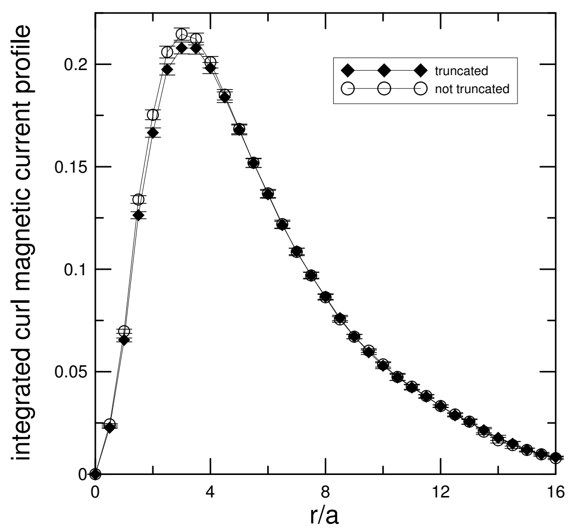

FIG. 12: The effect of truncation of monopole loops, keeping only the percolating cluster.

Finally we give a result that is disjoint from the body of this paper. We looked at monopole loops using the standard DT definition, i.e. constructed using $F^{(1)}_{\mu\nu}$. In this case there is a distinction between connected and disconnected loops. Earlier studies have shown that there is a single large connected percolating cluster of monopole currents that dominates confinement physics[14–17]. There is a sharp distinction between this percolating cluster and the large number of very small loops.

We confirm this result in our configurations and find that the big cluster contains $\sim 40\%$ of the current. We compared our calculations of the current with the truncated version, dropping $\sim 60\%$ of the contribution to the current coming from the small loops. Our results are shown in Fig. 12. This gives the integrated curl $\int_{r'<r} \mathrm{curl}\boldsymbol{J} \cdot d\boldsymbol{a'}$ as a function of r. The

truncation has a very minor effect which means that it has a minor effect in determining the parameters of the vortex.

V. SUMMARY AND CONCLUSIONS

Within the context of Abelian projected $SU(2)$, a dual Abrikosov vortex is a definitive signal of confinement. We argue here that this approach removes known systematic errors to the numerical evidence for such a structure. We point out that the chromoelectric flux should be defined through a Ward-Takahashi identity. Once this definition is fixed, then Maxwell's equations for expectation values of current require a consistent definition of flux throughout. This leads to a magnetic current definition differing from the standard monopole DeGrand-Toussaint construction. All definitions give the same continuum limit. The one promoted here removes errors in determining the total flux in the vortex through the electric Maxwell equations and the profile of the vortex through the magnetic Maxwell equations. The mixing of definitions can result in significant errors in determining the vortex parameters as illustrated in Eqs.(32).

The Wilson loop source is an electric line current. We find that it induces a dynamical electric current in the same plane of the loop. The value of the current is comparable to other observables. This is absent in the dual GLH model of the vortex. As a consequence the single London parameter serves as a proportionality between flux and the curl of the current and as the penetration length of electric field. In our simulation, these take on distinct values. A fit of the simulation data to a dual GLH model would be compromised since one parameter would have to account for two different phenomena. Therefore we abandoned such a fit and estimated the parameters based on their expected exponential behavior.

Normally one describes the physics of dual superconductivity with two parameters. However we claim there are three Λ_d, λ_d, and ξ_d. The first is very noisy and poorly defined, crudely $\Lambda_d/a \approx 1.0 \pm 0.5$. The second we obtained a stable value $\lambda_d/a = 1.83(4)$. However this value is dominated by the fit for small R and T and we presume that it contains a systematic error. The parameter ξ_d is inherently more difficult to measure than λ_d since it involves the second derivative of the flux. Our method of analysis is applicable to the case in which $\lambda_d > \xi_d$ as we find for a range of R's and T's. However for the extrapolated values, this inequality is reversed. The proper cautious conclusion is that these two values seem to be driven toward each other in the extrapolation.

Finally we showed that the truncation of a conventional monopole current to the largest cluster has a very small effect on the expectation value of the current in the presence of a source.

Acknowledgments

Work supported in part by the U.S. Department of Energy under grant no. DE-FG05-01 ER 40617. Our numerical computations were performed in part on the Louisiana State University Intel Xeon cluster, SuperMike, that is operated by LSU Center for Computation and Technology (CCT).

APPENDIX A: GINZBURG-LANDAU-HIGGS MODEL

The Higgs model has vortex solutions and we review here results that we use in the model-independent analysis. (We make a dual transformation in the end for the purpose of modeling.) Consider the Higgs action for a compact gauge field

$$\begin{aligned} S \;=\; & -\frac{a^4}{2e^2} \sum_{\boldsymbol{n},\mu\nu} \left\{\cos\left(\Delta^+_\mu \theta_\nu(\boldsymbol{n}) - \Delta^+_\nu \theta_\mu(\boldsymbol{n})\right) - 1\right\} \\ & + \sum_{\boldsymbol{n},\mu} \left|\Phi(\boldsymbol{n}) - e^{i\theta_\mu(\boldsymbol{n})}\Phi(\boldsymbol{n}+\mu)\right|^2 \\ & + \lambda^{(4)} \sum_{\boldsymbol{n}} \left(\left|\Phi(\boldsymbol{n})\right|^2 - v^2\right)^2 . \end{aligned}$$

The Euler-Lagrange equation for the gauge field is

$$\sum_\mu \frac{1}{a} \Delta^-_\mu F_{\alpha\mu}(\boldsymbol{m}) = J^{(e)}_\alpha(\boldsymbol{m})$$

where

$$F_{\alpha\mu} \equiv \frac{1}{ea^2} \sin\theta_{\alpha\mu}, \qquad \theta_{\alpha\mu} = \Delta^+_\alpha \theta_\mu - \Delta^+_\mu \theta_\alpha,$$

and

$$\begin{aligned} J^{(e)}_\alpha(\boldsymbol{m}) \;=\; & -\frac{ie}{a^3}\left(\Phi(\boldsymbol{m}) e^{-i\theta_\alpha(\boldsymbol{m})} \Phi^*(\boldsymbol{m}+\alpha)\right. \\ & \left. -\Phi^*(\boldsymbol{m}) e^{i\theta_\alpha(\boldsymbol{m})} \Phi(\boldsymbol{m}+\alpha)\right) . \end{aligned} \tag{A1}$$

The Euler-Lagrange equation for the Higgs field is

$$D^+_\alpha D^-_\alpha \Phi(\boldsymbol{m}) = 2\lambda^{(4)} \left(\Phi(\boldsymbol{m})\Phi^*(\boldsymbol{m}) - v^2\right) \Phi(\boldsymbol{m}) \tag{A2}$$

where we the LHS is the covariant lattice Laplacian

$$\begin{aligned} D^+_\alpha D^-_\alpha \Phi(\boldsymbol{m}) \; \equiv \; \sum_\mu \; & \Big(e^{+i\theta_\mu(\boldsymbol{m})}\Phi(\boldsymbol{m}+\mu) \\ & +e^{-i\theta_\mu(\boldsymbol{m}-\boldsymbol{\mu})}\Phi(\boldsymbol{m}-\boldsymbol{\mu}) - 2\Phi(\boldsymbol{m})\Big). \end{aligned}$$

The electric current is conserved by virtue of Eq.(A2)

$$\frac{1}{a}\Delta^-_\alpha J^{(e)}_\alpha(\boldsymbol{m}) \; = \; 0.$$

The magnetic current is defined by

$$J^{(m)}_\alpha \; = \; -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu} \; \frac{1}{a}\Delta^+_\beta F_{\mu\nu}, \tag{A3}$$

which vanishes in the approximation that $\sin x \approx x$. Conservation of the magnetic current is kinematical, not relying on a dynamical relation.

$$\frac{1}{a}\Delta^+_\alpha J^{(m)}_\alpha \; = \; 0.$$

The London relation follows directly from assuming spontaneous gauge symmetry breaking via a constrained Higgs field

$$\begin{aligned} \Phi(\boldsymbol{m}) \; &= \; \rho(\boldsymbol{m})e^{i\chi(\boldsymbol{m})} \\ &= \; ve^{i\chi(\boldsymbol{m})} \end{aligned} \tag{A4}$$

where $\rho(\boldsymbol{m})$ is constant in $\boldsymbol{m}$ which should hold deep inside a type II superconductor and is constant everywhere in an "extreme type II" superconductor. Take the curl of Eq.(A1) with a forward difference operator and use Eq.(A4):

$$\begin{aligned} & \epsilon_{\lambda\tau\beta\alpha}\Delta^+_\beta J^{(e)}_\alpha(\boldsymbol{m}) = -\frac{ie}{a^3}\epsilon_{\lambda\tau\beta\alpha}\Delta^+_\beta \times \\ & \left\{\Phi(\boldsymbol{m})e^{-i\theta_\alpha(\boldsymbol{m})}\Phi^*(\boldsymbol{m}+\alpha) \right. \\ & \qquad \left. -\Phi^*(\boldsymbol{m})e^{i\theta_\alpha(\boldsymbol{m})}\Phi(\boldsymbol{m}+\alpha)\right\}, \end{aligned}$$

$$\begin{aligned} & \Delta^+_1 J^{(e)}_2 - \Delta^+_2 J^{(e)}_1 = -\frac{2ev^2}{a^3} \times \\ & \left\{\Delta^+_1 \sin\left(\theta_2 + \Delta^+_2\chi\right) - \Delta^+_2 \sin\left(\theta_1 + \Delta^+_1\chi\right)\right\}. \end{aligned} \tag{A5}$$

Let $\sigma = 1$ and $\rho = 2$ in Eq.(10)

$$\Delta_1^+ J_2^{(e)} - \Delta_2^+ J_1^{(e)} =$$
$$-\frac{1}{ea^3}\Delta_\mu^- \Delta_\mu^+ \sin\theta_{12} + \left(\Delta_3^- J_4^{(m)} - \Delta_4^- J_3^{(m)}\right). \tag{A6}$$

By refining the mesh, small a, one can expand $\sin x \approx x$ and the leading term gives the familiar result that the magnetic current vanishes.

$$J_\alpha^{(m)} = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\, \Delta_\beta^+ \left\{\Delta_\mu^+ \theta_\nu - \Delta_\nu^+ \theta_\mu\right\} = 0.$$

Eqs.(A5,A6) become

$$\begin{aligned}\left(\frac{\Delta_1^+}{a} J_2^{(e)} - \frac{\Delta_2^+}{a} J_1^{(e)}\right) &= -\frac{\Delta_\mu^-}{a}\frac{\Delta_\mu^+}{a}\left(\frac{\theta_{12}}{ea^2}\right) \\ &= -\frac{2e^2 v^2}{a^2}\left(\frac{\theta_{12}}{ea^2}\right).\end{aligned}$$

We can identify the London penetration depth or equivalently the photon mass

$$\lambda^2 = m_\gamma^{-2} = \frac{a^2}{2e^2 v^2}.$$

A vortex solution follows similarly from Eqs.(A1, A4). We assume that the London penetration depth is larger than the coherence length and consequently Eqs.(A4) holds sufficiently far in the transverse direction from the vortex.

$$J_\mu^{(e)} = \frac{2ev^2}{a^3}\sin\left(\theta_\mu + \Delta_\mu^+ \chi\right).$$

For a sufficiently fine mesh

$$J_\mu^{(e)} = \frac{2ev^2}{a^3}\left(\theta_\mu + \Delta_\mu^+ \chi\right).$$

Consider a sum over a closed path encircling the axis of the vortex, C, at large transverse distances where $J_\mu^{(e)}$ is exponentially small. Further assume that $\chi = \phi$, where ϕ is the azimuthal angle. Then

$$\begin{aligned}\sum_C \theta_\mu &= 2\pi \\ \oint ea\boldsymbol{A}\cdot\frac{d\ell}{a} &= 2\pi \\ \oint \boldsymbol{A}\cdot d\boldsymbol{\ell} &= \frac{2\pi}{e} \\ \int \boldsymbol{B}\cdot d\boldsymbol{a} &= \frac{2\pi}{e},\end{aligned}$$

giving the quantization of magnetic flux.

We first take the dual of these results for the London relation and the vortex quantization and use them in our interpretation of large transverse distances in the simulation data.

APPENDIX B: FLUX IN THE $U(1)$ GAUGE THEORY

Consider

$$Z_W(\epsilon_\mu(\boldsymbol{m})) = \int [d\theta] \sin\theta_W \exp\left(\beta S\right),$$

$$S = \sum_{n,\ \mu>\nu} \left(\cos\theta_{\mu\nu}(\boldsymbol{n}) - 1\right), \qquad \beta = \frac{1}{e^2}.$$

The subscript of $Z_W(\epsilon_\mu(\boldsymbol{m}))$ refers to the incorporation of the source into the partition function and the argument is a variable defined as the shift of one particular link, $\theta_\mu(\boldsymbol{m}) \to \theta_\mu(\boldsymbol{m}) + \epsilon_\mu(\boldsymbol{m})$. This translation can be transformed away since the measure is invariant under such an operation. Therefore

$$\begin{aligned}
\delta Z_W &= \int [d\theta] \sin\theta_W \exp(\beta S)\Big|_{\theta_\mu \to \theta_\mu + \epsilon_\mu} \\
&\quad - \int [d\theta] \sin\theta_W \exp(\beta S) \\
&= \epsilon_\mu \int [d\theta] \left(\delta_\mu(\boldsymbol{m}) \cos\theta_W + \sin\theta_W \frac{1}{e^2}\frac{\partial S}{\partial\theta_\mu}\right) \times \\
&\qquad\qquad \exp\left(\beta S\right) \\
&= 0,
\end{aligned} \tag{B1}$$

where $\delta_\mu(\boldsymbol{m}) = \pm 1, 0$. It is $+1$ if the shifted link coincides with a loop link and oriented in the same direction, -1 if oriented oppositely, and 0 if the link is orthogonal or not located at a site of the loop. This is the static electric current generated by the Wilson loop with a normalization of unity.

$$\delta_\mu(\boldsymbol{m}) = \widehat{J}^e_\mu(\boldsymbol{m}). \tag{B2}$$

Next evaluate the derivative of S

$$\frac{\partial S}{\partial\theta_\mu(\boldsymbol{m})} = -\Delta^-_\nu \sin\theta_{\mu\nu}(\boldsymbol{m}). \tag{B3}$$

Using the definition, Eq.(13), we can see that Eq.(B1) is the form of a Maxwell equation for averages.

$$\frac{1}{e^2}\Delta_\nu^- \left\langle \widehat{F}^{(2)}_{\mu\nu} \right\rangle_W \;=\; \widehat{J}^e_\mu(\boldsymbol{m}), \tag{B4}$$

where

$$\langle \cdots \rangle_W \;=\; \frac{\langle \sin\theta_W \cdots \rangle}{\langle \cos\theta_W \rangle}.$$

(The conventional normalization of field strength and electric current is given in Sec. III F above.) Since the charged line in a Wilson loop is closed the electric current is conserved. The local statement of conservation is

$$0 = \Delta_\mu^- \Delta_\nu^- \left\langle \widehat{F}^{(2)}_{\mu\nu} \right\rangle_W = \Delta_\mu^- \widehat{J}^e_\mu.$$

It is straightforward to verify that the lattice averages on LHS of Eq.(B4) gives $-1, 0, +1$ depending on its position and orientation with respect to links in the Wilson loop.

For an alternative definition such as $\widehat{F}^{(1)}_{\mu\nu}$ the LHS need not vanish off the Wilson loop nor give ± 1 on the Wilson loop and hence would introduce an error in the source current.

APPENDIX C: $U(1)$ FLUX IN THE $SU(2)$ THEORY IN THE MAXIMAL ABELIAN GAUGE

We restrict our attention to the maximal Abelian gauge defined as a local maximum of

$$R \;=\; \sum_{n,\mu} \mathrm{tr}\left\{ \sigma_3 U_\mu(n) \sigma_3 U_\mu^\dagger(n) \right\},$$

over the set of gauge transformations $\left\{ g(m) = e^{i\alpha_i(m)\sigma_i} \right\}$, $U \longrightarrow U^g$. Taking U to be the stationary value, the stationary condition is given by

$$F_{jn}[U] \;=\; \left.\frac{\partial R[U^g]}{\partial \alpha_j(n)}\right|_{\alpha=0} = 0.$$

The second derivatives entering in the Jacobian are given by

$$M_{jn;im}(U) \;=\; \left.\frac{\partial^2 R[U^g]}{\partial \alpha_j(n) \partial \alpha_i(m)}\right|_{\alpha=0}.$$

The partition function is

$$Z_W^{g.f.}(\epsilon^3_\mu(\boldsymbol{m})) \;=\; \int [dU]\,\frac{1}{2}Tr[i\sigma_3 U_W(\boldsymbol{n})] \times \\ \exp{(\beta S)}\prod_{jn}\delta(F_{jn}[U])\;\Delta_{FP}, \tag{C1}$$

where the Faddeev-Popov Jacobian is

$$\Delta_{FP} \;=\; \det|M_{jn;im}(U)|.$$

An infinitesimal shift in this partition function has the added complication that it violates the gauge condition. This can be corrected by an infinitesimal accompanying gauge transformation. Thus the shift in one link affects all links. However experience has shown that the effect drops off rapidly with distance from the shifted link.

The derivative of the partition function Eq.(C1) with respect to $\epsilon^3_\mu(\boldsymbol{m})$ gives

$$\begin{aligned} 0 \;=\; & \int [dU]\left\{\delta_\mu(\boldsymbol{m})\frac{1}{2}Tr[U_W(\boldsymbol{n})]\right\}\exp{(\beta S)} \times \\ & \prod_{jn}\delta(F_{jn}[U])\Delta_{FP} \\ + & \int [dU]\left\{\beta\frac{1}{2}Tr[i\sigma_3 U_W(\boldsymbol{n})]\frac{\partial S}{\partial\epsilon^3_\mu(\boldsymbol{m})}\right\}\exp{(\beta S)} \times \\ & \prod_{jn}\delta(F_{jn}[U])\Delta_{FP} \\ + & \int [dU]\frac{1}{2}Tr[i\sigma_3 U_W(\boldsymbol{n})]\exp{(\beta S)} \times \\ & \frac{\partial}{\partial\epsilon^3_\mu(\boldsymbol{m})}\left\{\prod_{jn}\delta(F_{jn}[U])\Delta_{FP}\right\}. \end{aligned} \tag{C2}$$

The third integral contains terms in the Ward Takahashi identity coming from the gauge fixing including ghost contributions.

We can cast this into the form of the electric Maxwell equations for averages as in the case of the $U(1)$ theory. However there are now more terms in the current. Starting with Eq.(16):

$$\begin{aligned} U_\mu(\boldsymbol{n}) \;=\; & \begin{pmatrix} C_\mu e^{i\theta_\mu} & S_\mu e^{i(\gamma_\mu-\theta_\mu)} \\ -S_\mu e^{-i(\gamma_\mu-\theta_\mu)} & C_\mu e^{-i\theta_\mu} \end{pmatrix}, \\ =\; & \begin{pmatrix} C_\mu & S_\mu e^{i\gamma_\mu} \\ -S_\mu e^{-i\gamma_\mu} & C_\mu \end{pmatrix}\begin{pmatrix} e^{i\theta_\mu} & 0 \\ 0 & e^{-i\theta_\mu} \end{pmatrix} \end{aligned}$$

In the Abelian projection factored form, the righthand factor contains the $U(1)$ photon, parameterized by θ. The lefthand factor contains the charged coset matter fields, parameterized by ϕ and γ. The transformation properties are well known and reviewed in DiCecio et.al.[38].

We consider an alternative separation into diagonal and off-diagonal parts which is needed in defining the flux.

$$
\begin{aligned}
U_\mu(\boldsymbol{n}) &= \begin{pmatrix} C_\mu e^{i\theta_\mu} & 0 \\ 0 & C_\mu e^{-i\theta_\mu} \end{pmatrix} + \\
&\quad \begin{pmatrix} 0 & S_\mu e^{i(\gamma_\mu-\theta_\mu)} \\ -S_\mu e^{-i(\gamma_\mu-\theta_\mu)} & 0 \end{pmatrix}, \\
&= D_\mu(\boldsymbol{n}) + O_\mu(\boldsymbol{n}).
\end{aligned}
$$

The off-diagonal part is the charged matter field $\Phi_\mu \equiv S_\mu e^{-i(\gamma_\mu-\theta_\mu)}$. and diagonal part includes the photon, $e^{i\theta_\mu}$, but also a neutral remnant of the matter field $\sqrt{1-|\Phi_\mu|^2}$ which $\rightarrow 1$ in the limit $a \rightarrow 0$.

To cast Eq.(C2) into the form of a current conservation law, we first consider the terms to zeroth order in O_μ. First the Wilson loop. Isolating the diagonal contributions gives

$$
U_W = D_W + \widetilde{U}_W,
$$

where D_W is the product of the diagonal parts

$$
\begin{aligned}
\frac{1}{2}Tr[D_W(\boldsymbol{n})] &= \left(\prod_W C_\mu(\boldsymbol{n})\right)\cos\theta_W, \\
\frac{1}{2}Tr[i\sigma_3 D_W(\boldsymbol{n})] &= -\left(\prod_W C_\mu(\boldsymbol{n})\right)\sin\theta_W.
\end{aligned}
$$

We adhere to the standard choice of an Abelian Wilson loop in which we drop any contributions due to off diagonal elements O_μ and further take the factors $C_\mu(\boldsymbol{n}) = 1$ giving

$$
\begin{aligned}
\frac{1}{2}Tr[U_W^{\text{Abelian}}(\boldsymbol{n})] &= \cos\theta_W, \\
\frac{1}{2}Tr[i\sigma_3 U_W^{\text{Abelian}}(\boldsymbol{n})] &= -\sin\theta_W.
\end{aligned}
$$

Second, consider the action. Write

$$
S = \sum_{n,\,\mu>\nu} \frac{1}{2}\mathrm{Tr}[D_{\mu\nu}(\boldsymbol{n})] + \widetilde{S},
$$

where $\widetilde{S}$ contains terms involving $O_\mu(\boldsymbol{n})$.

$$\begin{aligned}\frac{\partial(S-\widetilde{S})}{\partial\epsilon^3_\mu(\boldsymbol{m})} &= \sum_{\nu\neq\mu}\left[\frac{1}{2}\mathrm{Tr}\left\{-i\sigma_3(\boldsymbol{m})D_\mu(\boldsymbol{m})\times\right.\right.\\ &\quad \left. D_\nu(\boldsymbol{m}+\mu)D^\dagger_\mu(\boldsymbol{m}+\nu)D^\dagger_\nu(\boldsymbol{m})\right\}\\ &\quad +\frac{1}{2}\mathrm{Tr}\left\{-i\sigma_3(\boldsymbol{m})D_\mu(\boldsymbol{m})\times\right.\\ &\quad \left.\left. D^\dagger_\nu(\boldsymbol{m}+\mu-\nu)D^\dagger_\mu(\boldsymbol{m}-\nu)D_\nu(\boldsymbol{m}-\nu)\right\}\right].\end{aligned}$$

Since all matrices are diagonal we can simplify:

$$\begin{aligned}\frac{\partial(S-\widetilde{S})}{\partial\epsilon^3_\mu(\boldsymbol{m})} &= \sum_{\nu\neq\mu}\Delta^-_\nu\left[C_\mu(\boldsymbol{m})\times\right.\\ &\quad \left. C_\nu(\boldsymbol{m}+\mu)C_\mu(\boldsymbol{m}+\nu)C_\nu(\boldsymbol{m})\sin\theta_{\mu\nu}(\boldsymbol{m})\right].\end{aligned}$$

The quantity in square brackets is antisymmetric in $\mu\nu$ and we identify this as proportional to the field tensor.

$$\begin{aligned}\widehat{F}^{(3)}_{\mu\nu} &\equiv C_\mu(\boldsymbol{m})C_\nu(\boldsymbol{m}+\mu)C_\mu(\boldsymbol{m}+\nu)C_\nu(\boldsymbol{m})\times\\ &\quad \sin\theta_{\mu\nu}(\boldsymbol{m}).\end{aligned} \tag{C3}$$

Returning to the identity, Eq.(C2), and using the notation

$$\int[dU]\{\cdots\}\exp(\beta S)\prod_{jn}\delta(F_{jn}[U])\Delta_{FP} = \langle\cdots\rangle_{g.f.}$$

we obtain

$$\begin{aligned}0 &= \delta_\mu(\boldsymbol{m})\left\langle\cos\theta_W\right\rangle_{g.f.}-\beta\left\langle\sin\theta_W\Delta^-_\nu\widehat{F}^{(3)}_{\mu\nu}\right\rangle_{g.f.}\\ &\quad -\beta\left\langle\sin\theta_W\frac{\partial\widetilde{S}}{\partial\epsilon^3_\mu(\boldsymbol{m})}\right\rangle_{g.f.}\\ &\quad +\text{gauge fixing terms}+\text{ghost terms}.\end{aligned}$$

Rearranging terms as in the $U(1)$ case, we get

$$\begin{aligned}\beta\frac{\left\langle\sin\theta_W\Delta^-_\nu\widehat{F}^{(3)}_{\mu\nu}\right\rangle_{g.f.}}{\left\langle\cos\theta_W\right\rangle_{g.f.}} &= \delta_\mu(\boldsymbol{m})\\ &\quad -\beta\frac{\left\langle\sin\theta_W\frac{\partial\widetilde{S}}{\partial\epsilon^3_\mu(\boldsymbol{m})}\right\rangle_{g.f.}}{\left\langle\cos\theta_W\right\rangle_{g.f.}}-\frac{\text{g.f. \& ghosts}}{\left\langle\cos\theta_W\right\rangle_{g.f.}}\end{aligned} \tag{C4}$$

This give the result analogous to Eq. (B4):

$$\begin{aligned}\beta\Delta_\nu^- \left\langle F_{\mu\nu}^{(3)}\right\rangle_{W,g.f.} &= \widehat{J}_\mu^{\text{(e) Abelian Wilson loop}} + \\ &\left\langle \widehat{J}\mu^{\text{(e) matter fields}}\right\rangle_{W,g.f.} + \\ &\left\langle \widehat{J}\mu^{\text{(e) gauge fixing}}\right\rangle_{W,g.f.} + \\ &\left\langle \widehat{J}\mu^{\text{(e) ghosts}}\right\rangle_{W,g.f.} .\end{aligned} \tag{C5}$$

The 'Abelian Wilson loop' term is analogous to the above $U(1)$ case. The 'charged matter field' term arises from the off-diagonal elements of links in the action expression and would contribute without gauge fixing. The 'gauge fixing' term arises from the corrective gauge transformation that accompanies the shift of a link. The 'ghost' term arises from the shift and accompanying gauge transformation on the Faddeev-Popov determinant. See DiCecio et.al.[38] for a complete derivation of all terms.

It is important here to note that we are not interested in distinguishing the various contributions to the current in the present work. We are only interested in the total current and that can be obtained from the LHS of Eq.(17).

---

[1] T. Suzuki, Nucl. Phys. B (Proc. Suppl.) **30**, 176 (1993).

[2] M. N. Chernodub and M. I. Polikarpov, "Confinement, duality and nonperturbative aspects of QCD", Ed by P. van Baal, Plenum Press, p. 387, hep-th/9710205.

[3] R. W. Haymaker Lectures, Int. Sch of Physics, "Enrico Fermi", Varenna, Italy, 1995, "Varenna 1995, Selected topics in nonperturbative QCD" 175-201. hep-lat/9510035.

[4] M.I. Polikarpov, Nucl. Phys. (Proc. Suppl.) **53**, 134 (1997)

[5] A. Di Giacomo, Prog. Theor. Phys. Suppl. **131**, 161 (1998), hep-lat/9802008.

[6] M.N. Chernodub, F.V. Gubarev, M.I. Polikarpov, and A.I. Veselov, 1997 Yukawa Int. Seminar on "Non-perturbative QCD - Structure of QCD Vacuum -" (YKIS'97), Kyoto, Prog. Theor. Phys. Suppl. **131**, 309 (1998), hep-lat/9802036.

[7] R. W. Haymaker, Phys. Rep. **315**, 153 (1999).

[8] M.N. Chernodub, F.V. Gubarev, M.I. Polikarpov, and V.I. Zakharov, hep-lat/0103033.

[9] G. Ripka, Trento Lectures, 2003, Springer, Berlin (2004), 639, hep-ph/0310102.

[10] A. S. Kronfeld, M. Laursen, G. Schierholz, and U.-J. Wiese, Phys. Lett. B **198**, 516 (1987).

[11] T. Suzuki and I. Yotsuyanagi, Phys. Rev. D **42**, 4257 (1990).

[12] J. D. Stack, S. D. Neiman, and R. J. Wensley, Phys. Rev. D **50**, 3399 (1994).

[13] H. Shiba and T. Suzuki, Phys. Lett. B **333**, 461 (1994).

[14] A. Hart and M. Teper, Phys. Rev. D **58**, 014504 (1998).

[15] V.G. Bornyakov, M.N. Chernodub, F.V. Gubarev, M. I. Polikarpov, T. Suzuki, A.I. Veselov, and V.I. Zakharov, Phys. Lett. B **537**, 291 (2002), hep-lat/0103032.

[16] M.N. Chernodub and V.I. Zakharov, Nucl.Phys. B **669**, 233 (2003).

[17] M.N. Chernodub, Katsuya Ishiguro, Katsuya Kobayashi, and Tsuneo Suzuki, Phys. Rev. D **69**, 014509 (2004), hep-lat/0306001.

[18] L. Del Debbio, Di Giacomo, and G. Paffuti, Phys. Lett. B, **349**, 513 (1995).

[19] A. Di Giacomo, B. Lucini, L. Montesi, and G. Paffuti, Phys. Rev. D **61**, 034503 (2000); 034504 (2000).

[20] J.M. Carmona, M. D'Elia, A. Di Giacomo, B. Lucini, and G. Paffuti, Phys. Rev. D **64**, 114507 (2001).

[21] V. Singh, D. A. Browne, and R. W. Haymaker, Phys. Lett. B **306**, 115 (1993).

[22] Y. Matsubara, S. Ejiri, and T. Suzuki, Nucl. Phys. B (Proc. Suppl.), **34**, 176 (1994).

[23] Y. Peng and R. W. Haymaker, Phys. Rev. D **52**, 3030 (1995).

[24] P. Cea and L. Cosmai, Phys. Rev. D **52**, 5152 (1995).

[25] G. S. Bali, C. Schlichter, and K. Schilling, Prog. Theor. Phys. Suppl. **131**, 645 (1998).

[26] G. S. Bali, Talk 3rd International Conference on Quark Confinement and the Hadron Spectrum (Confinement III), Newport News, VA, 7-12 Jun 1998, hep-ph/9809351.

[27] F. V. Gubarev, E.-M Ilgenfritz, M. I. Polikarpov, and T. Suzuki, Phys. Lett. B **468**, 134 (1999).

[28] H. Suganuma, K. Amemiya, H. Ichie, H. Matsufuru, Y. Nemoto, and T.T. Takahashi, (Confinement 2000), Osaka Japan, hep-lat/047020.

[29] H. Suganuma, K. Amemiya, and H. Ichie, Nucl. Phys. (Proc. Suppl) **83**, 547 (2000), hep-lat/0407015.

[30] Y. Koma, M. Koma, E.-M. Ilgenfritz, T. Suzuki, and M.I. Polikarpov, Phys. Rev. D **68**, 094018 (2003), hep-lat/0302006.

[31] Y. Koma, M. Koma, E.-M. Ilgenfritz, and T. Suzuki, Phys. Rev. D **68**, 114504 (2003), hep-lat/0308008.

[32] V. A. Belavin, M. N. Chernodub, and M. I. Polikarpov, hep-lat/0403013.

[33] Y. Matsubara, S. Ilyar, T. Okude, K. Yotsuji, and T. Suzuki, Nucl. Phys. (Proc. Suppl.), **42**, 529 (1995).

[34] S. Ejiri, S. I. Kitahara, Y. Matsubara, T. Okude, T. Suzuki, and K. Yasuta, Nucl. Phys. (Proc. Suppl.) **47**, 322 (1996).

[35] A. J. van der Sijs, Nucl. Phys. (Proc. Suppl.) **73**, 548 (1999).

[36] S. Fujimoto, S. Kato, T. Suzuki, and T. Tsunemi, Prog. Theor. Phys. Suppl. **138**, 36 (2000).

[37] B. S. Bali, V. Bornyakov, M. Müuller-Preussker, and K. Schilling, Phys. Rev. D **54**, 2863 (1996).

[38] G. DiCecio, A. Hart, and R. Haymaker, Phys. Lett. B **441**, 319 (1998).

[39] M. Zach, M. Faber, W. Kainz, and P. Skala Phys. Lett. B **358**, 325 (1995).

[40] T. Matsuki and R.W. Haymaker, Ed. by Suganuma et al, Wako 2003, Color Confinement and Hadrons in Quantum Chromodynamics, 21-24 July, 2003 World Scientific, Singapore, 2004, p60-71; Lattice 2003, Tsukuba, Nucl. Phys. Proc. Suppl. **129**, 641 (2004), hep-lat/03310017.

[41] T. A. DeGrand and D. Toussaint, Phys. Rev. D **22**, 2478 (1980).

[42] G. Poulis, Phys. Rev. D **54**, 6974 (1996).

Phys. A **670** (2000), hep-lat/0407017. T. T. Takahashi, Nucl. Phys.