

Consistency of lattice definitions of $U(1)$ flux in Abelian projected $SU(2)$ gauge theory*

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We reexamine the dual Abrikosov vortex under the requirement that the lattice averages of the fields satisfy exact Maxwell equations [ME]. The electric ME accounts for the total flux and the magnetic ME determines the shape of the confining string. This leads to unique and consistent definitions of flux and electric and magnetic currents at finite lattice spacing. The resulting modification of the standard DeGrand-Toussaint construction gives a magnetic current comprised of smeared monopoles.

In Abelian projected $SU(2)$ lattice gauge theory in the maximal Abelian gauge the existence of a dual Abrikosov vortex between quark and antiquark has been well established[1–4]. Here we tighten up the above picture further by incorporating the lattice Ward-Takahashi identity derived from the residual $U(1)$ gauge symmetry[5]. This is an Ehrenfest relation for the expectation value of the fields and currents giving the electric Maxwell equations [ME] exactly at finite lattice spacing. This defines a unique lattice expression for the field strength or flux.

In the present work, we are examining the impact of this on the study of the dual Abrikosov vortex. The Ward-Takashi identity determines uniquely the lattice definition of all components of the flux. To be consistent, the magnetic ME must use the same definition. The standard DeGrand-Toussaint[6] [DT] definition of the magnetic current is based on a different definition of flux, resulting in inconsistencies in the magnetic ME.

This consistency question is only relevant at finite lattice spacing and all these concerns go away in the continuum limit. However the finite lattice spacing effects are significant for the values of β that we often use for calculations.

Let us consider three definitions of field

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strength, all agreeing to lowest order in a . The first definition was used by DT to define monopoles:

$$\begin{aligned}\widehat{F}_{\mu\nu}^{(1)} &= \theta_\mu(\mathbf{m}) - \theta_\nu(\mathbf{m} + \nu) \\ &\quad - \theta_\nu(\mathbf{m}) + \theta_\nu(\mathbf{m} + \mu) - 2\pi n_{\mu\nu}, \\ &\equiv \theta_{\mu\nu}(\mathbf{m}) - 2\pi n_{\mu\nu}, \\ &= \bar{\theta}_{\mu\nu}(\mathbf{m}),\end{aligned}$$

where $\theta_\mu(\mathbf{m})$ refers to the $U(1)$ link angle in the domain $-\pi < \theta_\mu < +\pi$. The integers $n_{\mu\nu}$ are determined by requiring that $-\pi < \bar{\theta}_{\mu\nu} < +\pi$. ($\bar{\theta}_{\mu\nu}$ is a periodic function of $\theta_{\mu\nu}$ with period 2π , i.e. it is a “sawtooth function”.)

The second and third gives the exact electric ME for lattice averages

$$\Delta_\mu^- \langle \widehat{F}_{\mu\nu}^{(i)} \rangle_W = \langle \widehat{J}_\mu^{(i)} \rangle_W, \quad i = 2, 3, \quad (1)$$

where

$$\langle \dots \rangle_W = \frac{\langle \dots e^{i\theta_W} \rangle}{\langle e^{i\theta_W} \rangle}.$$

For a pure $U(1)$ theory with a Wilson action, and for an Abelian projected $SU(2)$ theory with Wilson action respectively :

$$\widehat{F}_{\mu\nu}^{(2)} = \sin \theta_{\mu\nu}. \quad (2)$$

$$\widehat{F}_{\mu\nu}^{(3)} = C_\mu(\mathbf{m})C_\nu(\mathbf{m} + \mu)C_\mu(\mathbf{m} + \nu)C_\nu(\mathbf{m}) \times \sin \theta_{\mu\nu}. \quad (3)$$

where the C factors are associated with the matter field in the Abelian projection, as explained

below. Here quantities with \sim mean those which appear in the lattice numerical calculation without appending factors of e and a .

To derive Eqn.(1) for the $U(1)$ case consider

$$Z_W(\epsilon_\mu(\mathbf{m})) = \int [d\theta] \sin \theta_W \exp(\beta S).$$

This is invariant under the shift of any link angle

$$\theta_\mu(\mathbf{m}) \rightarrow \theta_\mu(\mathbf{m}) + \epsilon_\mu(\mathbf{m})$$

which leads to Eqn.(1)[7]. For the $SU(2)$ case[5] we use instead the fact that the Haar measure is invariant under multiplication by an infinitesimal group element $1 + i\epsilon_3\sigma_3$. The definition of flux that emerges involves the diagonal elements of the $SU(2)$ links

$$\hat{F}_{\mu\nu}^{(3)} = \frac{1}{2i} T^r(\sigma_3) D_\mu(\mathbf{m}) D_\nu(\mathbf{m} + \mu) \times D_\nu^\dagger(\mathbf{m} + \nu) D_\mu^\dagger(\mathbf{m}),$$

where

$$D_\mu(\mathbf{n}) = \begin{pmatrix} C_\mu e^{i\theta_\mu} & 0 \\ 0 & C_\mu e^{-i\theta_\mu} \end{pmatrix}.$$

Working in a fixed gauge significantly complicates the derivation since the shift of a link takes one out of the gauge. A compensating infinitesimal gauge transformation restores the gauge.

Having defined a unique flux $\hat{F}_{\mu\nu}^{(i)}$ through the electric ME, the magnetic ME is

$$\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \Delta_\nu^+ \hat{F}_{\rho\sigma}^{(i)} = \hat{J}_\mu^{(i)} \quad i = 2, 3$$

However the standard DT definition of current is

$$\hat{J}_\mu^{m(1)} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \Delta_\nu^+ \hat{F}_{\rho\sigma}^{(1)}$$

and hence if we use the conventional $\hat{F}^{(1)}$ to define the monopole current, and $F^{(2)}$ or $\hat{F}^{(3)}$ respectively for $U(1)$ and $SU(2)$ theories to get an exact expression for flux in the confining string, then the magnetic ME is violated.

The solution is to relax the requirement that we use the DT monopole definition and use $F^{(2)}$ or $\hat{F}^{(3)}$ instead when defining monopoles. A simple configuration for the $U(1)$ case ($F^{(2)}$) will illustrate the effect. Consider a single DT monopole

with equal flux out of the six faces of the cube. Then the ratio of the $\hat{F}^{(2)}$ flux out of this cube compared to the DT $\hat{F}^{(1)}$ flux gives

$$\frac{6 \sin(2\pi/6)}{6(2\pi/6)} \approx 0.83.$$

Since the current is conserved, the balance is made up by magnetic charge in the neighboring cells. On a large surface the total flux is the same for the two definitions.

The electric ME determines the total flux in the confining string and the magnetic ME determines the transverse shape. Further the latter enters directly in the determination of the London penetration length, λ_d . To see this let us consider the classical Higgs theory which we use to model the simulation data. The dual field is given by

$$\hat{G}_{\mu\nu}(\mathbf{m}) = \Delta_\mu^- \theta_\nu^{(d)}(\mathbf{m}) - \Delta_\nu^- \theta_\mu^{(d)}(\mathbf{m}),$$

where $\theta_\mu^{(d)}(\mathbf{m})$ is a dual link variable. Let us choose to break the gauge symmetry spontaneously through a constrained Higgs field.

$$\Phi(\mathbf{m}) = v\rho(\mathbf{m})e^{i\chi(\mathbf{m})}, \quad \rho(\mathbf{m}) = 1.$$

Under these conditions the magnetic current simplifies to

$$\hat{J}_\mu^m(\mathbf{m}) = 2e_m^2 v^2 \sin\{\theta_\mu(\mathbf{m}) + \chi_\mu(\mathbf{m} + \mu) - \chi_\mu(\mathbf{m})\}.$$

where e_m is the magnetic coupling. The magnetic ME is

$$\Delta_\mu^+ \hat{G}_{\mu\nu} = \hat{J}_\nu^m. \quad (4)$$

For small $\theta^{(d)}$ it is easy to see that there is a London relation of the form

$$\hat{G}_{\mu\nu}(\mathbf{m}) = \frac{1}{2e_m^2 v^2} (\Delta_\mu^- \hat{J}_\nu^m(\mathbf{m}) - \Delta_\nu^- \hat{J}_\mu^m(\mathbf{m})). \quad (5)$$

Taking the confining string along the 3 axis and choosing $\mu = 1$ and $\nu = 2$ we see that the profile of the third component of curl of the magnetic current must match the third component of the electric flux profile. This assumes an infinite Higgs mass M_H . With a finite mass there is a transition region of size $\sim 1/M_H$ in the core of the vortex but the above London relation must hold sufficiently far outside the core.

Combining Eqns.(4) and (5) we get the relation

$$(1 - \lambda_2^2 \Delta_\mu^+ \Delta_\mu^-) \langle \widehat{E}_3(m) \rangle_W = 0, \lambda_d^{-1} = e_{m^0} v \sqrt{2}$$

The corresponding equations in the simulation of the SU(2) theory must also be satisfied in order to arrive at this correct expression for λ_d and hence the importance of our definitions.

We generated 208 configurations on a 32^4 lattice at $\beta = 2.5115$. The maximal Abelian gauge fixing used over-relaxation. Fig.(1) shows the profile of the electric flux corresponding to $\widehat{F}^{(2)}$ and $\widehat{F}^{(3)}$. Fig.(2) shows the profile of the theta component of the magnetic current corresponding to $\widehat{F}^{(1)}$, $\widehat{F}^{(2)}$ and $\widehat{F}^{(3)}$.

In summary, we showed that consistency requires one use the same definition of flux throughout. If, for example, one uses $\widehat{F}^{(3)}$ definition of electric field (bottom graph in Fig.(1)) in order to account correctly for the total flux but then uses the DT definition of current (top graph in Fig.(2)) we would then incur errors of $\sim 40\%$.

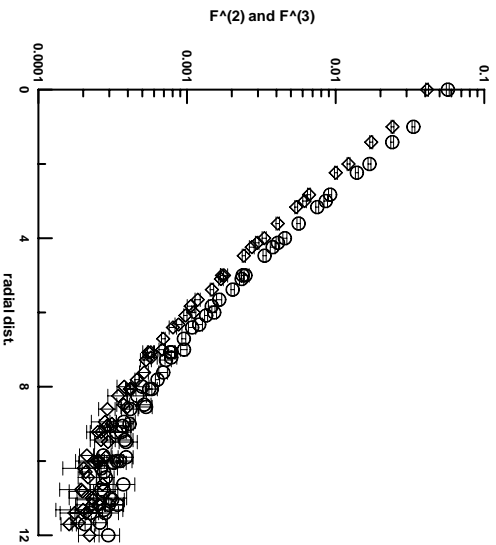


Figure 1. Profile of the electric field (highest to lowest) $\widehat{E}_3^{(2)}$ (circle) and $\widehat{E}_3^{(3)}$ (diamond) on the mid-plane between q and \bar{q} separated by 13a.

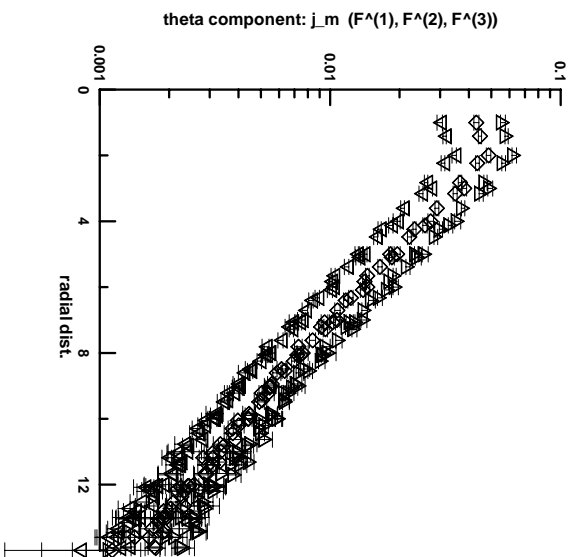


Figure 2. Profile of the theta component of the magnetic current on the mid-plane between q and \bar{q} separated by 13a based on (highest to lowest) $\widehat{F}_{\mu\nu}^{(1)}$ (triangles), $\widehat{F}_{\mu\nu}^{(2)}$ (diamonds) and $\widehat{F}_{\mu\nu}^{(3)}$ (inverted triangles).

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